ELSEVIER Journal of Geometry and Physics 31 (1999) 195–208

JOURNAL OF GEOMETRY AND PHYSICS

Integral formulas for compact space-like hypersurfaces in de Sitter space: Applications to the case of constant higher order mean curvature

Juan A. Aledo [a], Luis J. Alías [b,*], Alfonso Romero [c]

[a] *Departamento de Matemáticas, Universidad de Castilla-La Mancha, 02071 Albacete, Spain*
[b] *Departamento de Matemáticas, Universidad de Murcia, 30100 Espinardo, Murcia, Spain*
[c] *Departamento de Geometría y Topología, Universidad de Granada, 18071 Granada, Spain*

Received 20 October 1998

Abstract

In this paper we develop some integral formulas for compact space-like hypersurfaces in de Sitter space $\mathbf{S}_1^{n+1}$ and apply them in order to characterize the totally umbilical round spheres of $\mathbf{S}_1^{n+1}$ as the only compact space-like hypersurfaces with constant higher order mean curvature under some appropriate hypothesis. In particular, for hypersurfaces contained in the chronological future (or past) of an equator of $\mathbf{S}_1^{n+1}$ we prove that the only compact space-like hypersurfaces with a constant higher order mean curvature are the totally umbilical round spheres.

Subj. Class.: Differential property; General relativity
1991 MSC: Primary 53C42, Secondary 53B30 and 53C50
Keywords: de Sitter space; Space-like hypersurfaces; Higher order mean curvatures; Minkowski formulas

1. Introduction

The study of space-like hypersurfaces in Lorentzian space–times has been recently of substantial interest from both physical and mathematical points of view. From the physical one, that interest became clear when Lichnerowicz [10] showed that the Cauchy problem of the Einstein equation with initial conditions on a space-like hypersurface with vanishing mean extrinsic curvature has a particularly nice form, reducing to a linear differential system

* Corresponding author. Tel.: +34-968-364180; fax: +34-968-364182; e-mail: ljalias@fcu.um.es

PII: S0393-0440(99)00008-X

of first-order and to a non-linear second-order elliptic differential equation. We also refer the reader to the survey papers [4,11], and references therein for other reasons justifying their importance in general relativity.

From a mathematical point of view, space-like hypersurfaces are also interesting because of their Bernstein-type properties. When the ambient space–time is the de Sitter space $\mathbf{S}_1^{n+1}$, Goddard [6] conjectured that the only complete space-like hypersurfaces in $\mathbf{S}_1^{n+1}$ with constant mean curvature H should be the totally umbilical ones. Although this conjecture has turned out to be false in its original statement, it has motivated a great deal of work trying to find some positive answer under appropriate additional hypotheses. For instance, in [1] Akutagawa showed that Goddard's conjecture is true when $0 \leq H^2 \leq 1$ in the case $n = 2$, and when $0 \leq H^2 < 4(n-1)/n^2$ in the case $n \geq 3$. Later, Montiel [12] solved Goddard's problem in the compact case proving that the only compact space-like hypersurfaces in $\mathbf{S}_1^{n+1}$ with constant mean curvature are the totally umbilical round spheres (see also [14] for an alternative proof of both facts in the two-dimensional case).

On the other hand, Cheng and Ishikawa have recently shown that the totally umbilical round spheres are the only compact space-like hypersurfaces in de Sitter space with constant scalar curvature $S < n(n-1)$. Some other authors, such as Li [9] and Zheng [17,18], have also obtained interesting results related to the characterization of the totally umbilical round spheres as the only compact space-like hypersurfaces in $\mathbf{S}_1^{n+1}$ with constant scalar curvature.

The natural generalization of mean and scalar curvatures for a space-like hypersurface in de Sitter space are the rth mean curvatures H_r for $r = 1, \ldots, n$. Actually, H_1 is the mean curvature and H_2 is, up to a constant, the scalar curvature of the hypersurface (for the details, see Section 2). In this paper we will develop some integral formulas for compact space-like hypersurfaces in de Sitter space, which in analogy with the Euclidean case will be called Minkowski formulas. We will also obtain some applications of those integral formulas in order to characterize the totally umbilical round spheres in $\mathbf{S}_1^{n+1}$ as the only compact space-like hypersurfaces with constant higher order mean curvature under some appropriate hypothesis. For instance, we show (Theorem 3):

> The only compact space-like hypersurfaces in de Sitter space having H_r and H_{r+1} both constant, with $0 \leq r \leq n-2$, are the totally umbilical round spheres.

With respect to the case where only one rth mean curvature is constant, our main result is (Theorem 7):

> The only compact space-like hypersurfaces in de Sitter space with constant rth mean curvature H_r, $2 \leq r \leq n$, which are contained in the chronological future (or past) of an equator of $\mathbf{S}_1^{n+1}$ are the totally umbilical round spheres.

2. Preliminaries

Let $\mathbf{L}^{n+2}$ be the $(n+2)$-dimensional Lorentz–Minkowski space, $n \geq 2$, endowed with the Lorentzian metric tensor $\langle\, ,\rangle$ given by

$$\langle v , w \rangle = \sum_{i=1}^{n+1} v_i w_i - v_{n+2} w_{n+2},$$

and let $\mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$ be the $(n+1)$-dimensional unitary de Sitter space, i.e.,

$$\mathbf{S}_1^{n+1} = \{x \in \mathbf{L}^{n+2} : \langle x , x \rangle = 1\}.$$

As is well known, for $n \geq 2$ the de Sitter space $\mathbf{S}_1^{n+1}$ is the standard simply connected Lorentzian space form of constant sectional curvature one. A smooth immersion $\psi : M^n \longrightarrow \mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$ of an n-dimensional connected manifold M^n is said to be a space-like hypersurface if the induced metric via ψ is a Riemannian metric on M^n, which, as usual, is also denoted by $\langle\ ,\ \rangle$.

Throughout this paper we will deal with compact space-like hypersurfaces in de Sitter space. Observe that every such hypersurface $\psi : M^n \longrightarrow \mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$ is diffeomorphic to an n-sphere by means of the map $F = \Pi \circ \phi \circ \psi : M^n \longrightarrow \mathbf{S}^n$, where $\Pi : \mathbf{S}^n \times \mathbf{R} \to \mathbf{S}^n$ is the projection onto $\mathbf{S}^n$ and $\phi^{-1} : \mathbf{S}^n \times \mathbf{R} \longrightarrow \mathbf{S}_1^{n+1}$ is given by $\phi^{-1}(u, v) = ((\sqrt{1+v^2})u, v)$. Indeed, F is a local diffeomorphism, and the compactness of M^n and the simply connectedness of $\mathbf{S}^n$ imply that F is a global one. In particular, every compact space-like hypersurface in de Sitter space is orientable and there exists a time-like unit normal field N globally defined on M^n. We will refer to N as the Gauss map of the immersion and we will say that M^n is oriented by N.

Let $\psi : M^n \longrightarrow \mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$ be an immersed compact space-like hypersurface in de Sitter space $\mathbf{S}_1^{n+1}$, and let N be its Gauss map. In order to set up the notation, let us denote by ∇^o, $\bar{\nabla}$ and ∇ the Levi-Civita connections of $\mathbf{L}^{n+2}$, $\mathbf{S}_1^{n+1}$ and M^n, respectively. Then the Gauss and Weingarten formulas for M^n in $\mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$ are given, respectively, by

$$\nabla^o{}_X Y = \bar{\nabla}_X Y - \langle X , Y \rangle \psi = \nabla_X Y - \langle AX , Y \rangle N - \langle X , Y \rangle \psi, \tag{1}$$

and

$$A(X) = -\nabla^o_X N = -\bar{\nabla}_X N, \tag{2}$$

for all tangent vector fields $X, Y \in \mathcal{X}(M)$, where $A : \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$ stands for the shape operator of M^n in $\mathbf{S}_1^{n+1}$ with respect to N.

Associated to the shape operator of M^n there are n algebraic invariants, which are the elementary symmetric functions σ_r of its principal curvatures $k_1, \ldots, k_n$ given by

$$\sigma_r(k_1, \ldots, k_n) = \sum_{i_1 < \cdots < i_r} k_{i_1} \cdots k_{i_r}, \quad 1 \leq r \leq n.$$

The rth mean curvature H_r of the space-like hypersurface is then defined by

$$\binom{n}{r} H_r = (-1)^r \sigma_r(k_1, \ldots, k_n) = \sigma_r(-k_1, \ldots, -k_n).$$

When $r = 1$, $H_1 = -(1/n)\,\mathrm{tr}(A) = H$ is the mean curvature of M^n. The choice of the sign $(-1)^r$ in our definition of H_r is motivated by the fact that in that case the mean curvature vector is given by $\mathbf{H} = HN$. Therefore, $H(p) > 0$ at a point $p \in M^n$ if and only if $\mathbf{H}(p)$ is in the time-orientation determined by $N(p)$.

On the other hand, when $r = n$, $H_n = (-1)^n \det(A)$ defines the Gauss–Kronecker curvature of the space-like hypersurface, and for $r = 2$, H_2 is, up to a constant, the scalar curvature S of M^n. Indeed, the Ricci curvature of M^n is given by

$$\mathrm{Ric}(X, Y) = (n-1)\langle X, Y\rangle - \mathrm{tr}(A)\langle A(X), Y\rangle + \langle A(X), A(Y)\rangle,$$

for $X, Y \in \mathcal{X}(M)$, so that its scalar curvature is

$$S = \mathrm{tr}(\mathrm{Ric}) = n(n-1) - \mathrm{tr}(A)^2 + \mathrm{tr}(A^2) = n(n-1)(1 - H_2). \quad (3)$$

Observe that the characteristic polynomial of A can be written in terms of the H_r's as

$$\det(tI - A) = \sum_{r=0}^{n} \binom{n}{r} H_r t^{n-r}, \quad (4)$$

where $H_0 = 1$.

3. Integral formulas

In this section we will derive some general integral formulas for compact space-like hypersurfaces in de Sitter space. In order to do that, we will introduce the corresponding Newton transformations $T_r : \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$ arising from the shape operator A, which according to our definition of the rth mean curvatures are given by

$$T_r = \binom{n}{r} H_r I + \binom{n}{r-1} H_{r-1} A + \cdots + \binom{n}{1} H_1 A^{r-1} + A^r,$$

where I denotes the identity in $\mathcal{X}(M)$, or inductively,

$$T_0 = I \quad \text{and} \quad T_r = \binom{n}{r} H_r I + A T_{r-1}.$$

As a consequence of (4) it follows that $T_n = 0$. Observe that $T_r = (-1)^r \tilde{T}_r$, where $\tilde{T}_r$ is the rth Newton transformation defined by Reilly [15, Section 1]. The following algebraic properties of T_r can be found in [15] (see also [16, Section 4]),

$$\mathrm{tr}(T_r) = (n-r)\binom{n}{r} H_r, \quad (5)$$

$$\mathrm{tr}(AT_r) = -(r+1)\binom{n}{r+1} H_{r+1}, \quad (6)$$

where $0 \le r \le n-1$.

Let us consider, associated to each Newton transformation T_r, $0 \le r \le n-1$, the corresponding second-order differential operator L_r acting on $C^\infty(M)$ given by

$$L_r(u) = \mathrm{div}(T_r(\nabla u)).$$

In particular, when $r = 0$ the operator L_0 is nothing but the Laplacian operator of M^n. Using that $\nabla_X T_r$ is self-adjoint for any $X \in \mathcal{X}(M)$, an easy computation shows that

$$L_r(u) = \langle \operatorname{div}(T_r), \nabla u\rangle + \sum_{i=1}^{n} \langle T_r(\nabla_{E_i} \nabla u), E_i\rangle, \tag{7}$$

where $\{E_1, \ldots, E_n\}$ is a local orthonormal frame on M^n, and

$$\operatorname{div}(T_r) = \operatorname{tr}(\nabla T_r) = \sum_{i=1}^{n} (\nabla_{E_i} T_r)(E_i).$$

Let us now remark that $\operatorname{div}(T_r) = 0$.

Lemma 1. *The Newton transformations T_r are divergence-free.*

Proof. From the inductive definition of T_r we have

$$\sum_{i=1}^{n} (\nabla_{E_i} T_r)(E_i) = \binom{n}{r} \nabla H_r + \sum_{i=1}^{n} (\nabla_{E_i} A)(T_{r-1} E_i) + A\left(\sum_{i=1}^{n} (\nabla_{E_i} T_{r-1})(E_i)\right).$$

Using now

$$\operatorname{tr}(T_{r-1} \nabla_X A) = -\binom{n}{r} \langle \nabla H_r , X\rangle$$

(see, for instance, [16, Eq. (4.4)]), we obtain from the Codazzi equation

$$(\nabla_X A)(Y) = (\nabla_Y A)(X)$$

that

$$\sum_{i=1}^{n} (\nabla_{E_i} A)(T_{r-1} E_i) = -\binom{n}{r} \nabla H_r,$$

so that

$$\sum_{i=1}^{n} (\nabla_{E_i} T_r)(E_i) = A\left(\sum_{i=1}^{n} (\nabla_{E_i} T_{r-1})(E_i)\right).$$

An inductive argument implies then that

$$\sum_{i=1}^{n} (\nabla_{E_i} T_r)(E_i) = 0$$

for $r = 0, \ldots, n-1$. □

From Lemma 1, Eq. (7) becomes

$$L_r(u) = \sum_{i=1}^{n} \langle T_r(\nabla_{E_i} \nabla u), E_i\rangle = \sum_{i=1}^{n} T_r(E_i, E_j) \nabla^2 u(E_i, E_j). \tag{8}$$

In particular, when $r = 1$ the operator L_1 agrees (up to the sign) with the operator $\square$, which was introduced by Cheng and Yau [3]. The operator $\square$ has been recently used for the study of space-like hypersurfaces with constant scalar curvature in de Sitter space by several authors like Zheng [17,18], Li [9], and Cheng and Ishikawa [2].

Let $a \in \mathbf{L}^{n+2}$ be a fixed arbitrary vector, and consider the height function $\langle a , \psi \rangle$ defined on M^n. From (1) it is easy to see that its gradient is given by $\nabla \langle a , \psi \rangle = a^{\mathrm{T}}$, where

$$a^{\mathrm{T}} = a + \langle a , N \rangle N - \langle a , \psi \rangle \psi \tag{9}$$

is tangent to M^n. By taking covariant derivative in (9) and using (1) and (2), we obtain from $\nabla^{\mathrm{o}} a = 0$ that

$$\nabla_X a^{\mathrm{T}} = -\langle a , N \rangle A(X) - \langle a , \psi \rangle X \tag{10}$$

for $X \in \mathcal{X}(M)$. Therefore, using (5), (6) and (10) we obtain from (8) that

$$\begin{aligned} L_r(\langle a , \psi \rangle) &= -\langle a , \psi \rangle \operatorname{tr}(T_r) - \langle a , N \rangle \operatorname{tr}(AT_r) \\ &= -(n-r)\binom{n}{r} H_r \langle a , \psi \rangle + (r+1)\binom{n}{r+1} H_{r+1} \langle a , N \rangle \\ &= c_r(-H_r \langle a , \psi \rangle + H_{r+1} \langle a , N \rangle), \end{aligned} \tag{11}$$

where $c_r = (n-r)\binom{n}{r} = (r+1)\binom{n}{r+1}$. Integrating now (11) on M^n, the divergence theorem implies our integral formulas.

Theorem 2 (Minkowski formulas). *Let $\psi : M^n \longrightarrow \mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$ be a compact space-like hypersurface immersed into de Sitter space and let $a \in \mathbf{L}^{n+2}$ a fixed arbitrary vector. For each $r = 0, \ldots, n-1$, the following formula holds:*

$$\int_M (-H_r \langle a , \psi \rangle + H_{r+1} \langle a , N \rangle) \, \mathrm{d}V = 0,$$

where $\mathrm{d}V$ is the n-dimensional volume element of M^n with respect to the induced metric and the chosen orientation.

Our proof of Minkowski formulas here follows the ideas of Reilly [15] (see also [16]). In Appendix A, we will present another proof of these Minkowski formulas which uses the original ideas of Hsiung [8] in his proof of Minkowski formulas for compact hypersurfaces in Euclidean space (see also [13] for a more accessible modern treatment). We would like to include also that other proof in Appendix A because of its nice geometric feeling.

4. First applications

As a first application of Minkowski formulas, if the mean curvature H_1 is constant, multiplying by the constant H_1 the first Minkowski formula (for $r = 0$) and subtracting the second one (for $r = 1$), we obtain that

$$\int_M (H_1^2 - H_2)\langle a\,,N\rangle\,\mathrm{d}V = 0. \tag{12}$$

Observe that, by Cauchy–Schwarz inequality, we have

$$H_1^2 - H_2 = \frac{1}{n(n-1)}\left(\sum_{i=1}^{n} k_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} k_i\right)^2\right) \geq 0$$

equality holding only at umbilical points. Therefore, if we choose $a \in \mathbf{L}^{n+2}$ a unit time-like vector in the same time-orientation as N, then $\langle a\,,N\rangle \leq -1 < 0$ and from (12) we deduce that $H_1^2 - H_2 \equiv 0$ and the hypersurface must be totally umbilical. This provides us with a proof of the theorem given by Montiel [12].

Since $H_0 = 1$ by definition and the only compact space-like hypersurfaces in de Sitter space which are totally umbilical are the round spheres, the result by Montiel can be read as follows:

The only compact spacelike hypersurfaces in de Sitter space having H_0 and H_1 both constant are the totally umbilical round spheres.

The same argument as above can be used to prove the following generalization to any two consecutive rth mean curvatures.

Theorem 3. *The only compact space-like hypersurfaces in de Sitter space having H_r and H_{r+1} both constant, with $0 \leq r \leq n-2$, are the totally umbilical round spheres.*

Proof. Multiplying by the constant H_{r+1} the integral formula

$$\int_M (-H_r\langle a\,,\psi\rangle + H_{r+1}\langle a\,,N\rangle)\,\mathrm{d}V = 0,$$

and multiplying by the constant H_r the integral formula

$$\int_M (-H_{r+1}\langle a\,,\psi\rangle + H_{r+2}\langle a\,,N\rangle)\,\mathrm{d}V = 0,$$

we obtain, subtracting them, that

$$\int_M (H_{r+1}^2 - H_r H_{r+2})\langle a\,,N\rangle\,\mathrm{d}V = 0.$$

It is known (see [7, Theorem 55]) that

$$H_{r+1}^2 - H_r H_{r+2} \geq 0 \tag{13}$$

equality holding at umbilical points. Therefore, choosing $a \in \mathbf{L}^{n+2}$ a unit time-like vector in the same time-orientation as N, we deduce as above that the hypersurface must be a totally umbilical round sphere. □

5. Hypersurfaces with constant higher order mean curvature

From now on we will focus on the case where only one rth mean curvature is constant. Our objective is to find appropriate conditions under which the totally umbilical round spheres are the only compact space-like hypersurfaces in de Sitter space with H_r constant. For $r = 1$ this is the result given by Montiel [12]. When $r = 2$, since H_2 is (up to a constant) the scalar curvature of the hypersurface (see Eq. (3)), this corresponds to the case of constant scalar curvature. Li [9] has recently shown that when $n = 2$, the round spheres are the only compact space-like surfaces in $\mathbf{S}_1^3$ with constant scalar curvature, i.e., constant Gauss curvature. For the n-dimensional case, he has also shown that a compact space-like hypersurface in $\mathbf{S}_1^{n+1}$ with constant scalar curvature S satisfying $(n-1)(n-2) \leq S \leq n(n-1)$ must be a round sphere. Zheng [17] shows that if a compact space-like hypersurface in $\mathbf{S}_1^{n+1}$ with non-negative sectional curvature has constant scalar curvature S satisfying $S \leq n(n-1)$ then it must be a round sphere (see also [18] for a weaker version of this result). More recently, Cheng and Ishikawa have shown that the round spheres are the only compact space-like hypersurfaces in the de Sitter space with constant scalar curvature $S < n(n-1)$.

Let $\psi : M^n \longrightarrow \mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$ be a compact space-like hypersurface in de Sitter space with constant rth mean curvature H_r, $2 \leq r \leq n-1$. Let us consider the two following Minkowski formulas:

$$\int_M (-\langle a\,,\psi\rangle + H_1\langle a\,,N\rangle)\,\mathrm{d}V = 0,$$

and

$$\int_M (-H_r\langle a\,,\psi\rangle + H_{r+1}\langle a\,,N\rangle)\,\mathrm{d}V = 0.$$

If we multiply the first one by the constant H_r and subtract the second one, we obtain

$$\int_M (H_1H_r - H_{r+1})\langle a\,,N\rangle\,\mathrm{d}V = 0. \tag{14}$$

Choosing $a \in \mathbf{L}^{n+2}$ a unit time-like vector in the same time-orientation as N, then $\langle a\,,N\rangle \leq -1 < 0$. Therefore, if we were able to prove that $H_1H_r - H_{r+1} \geq 0$, with equality at the umbilical points, we could conclude that the hypersurface should be a totally umbilical round sphere. When $r = 2$ this follows from the assumption that H_2 is a positive constant or, equivalently (see Eq. (3)), that the scalar curvature S is constant and satifies $S < n(n-1)$. Indeed, by Cauchy–Schwarz inequality we know that $H_1^2 \geq H_2 > 0$, so that H_1 does not vanish on M^n, and by choosing the appropriate orientation, we may suppose that $H_1 > 0$ on M^n. Moreover, from (13) we also know that $H_2^2 - H_1H_3 \geq 0$, i.e.,

$$H_3 \leq \frac{H_2^2}{H_1}.$$

Therefore, we have

$$H_1H_2 - H_3 \geq H_1H_2 - \frac{H_2^2}{H_1} = \frac{H_2}{H_1}(H_1^2 - H_2) \geq 0,$$

with equality at the umbilical points. This provides us with another proof of the recent result given by Cheng and Ishikawa [2].

Theorem 4. *Let $\psi : M^n \longrightarrow \mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$ be a compact space-like hypersurface in de Sitter space, $n \geq 3$, with constant scalar curvature S satisfying $S < n(n-1)$. Then M^n is a totally umbilical round sphere.*

Moreover, one of the advantages of our approach is that it allows us to extend the result to the case of $r \geq 3$ in the following form.

Theorem 5. *Let $\psi : M^n \longrightarrow \mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$ be a compact space-like hypersurface in de Sitter space with constant rth mean curvature H_r, $2 \leq r \leq n-1$. If there exists a point $p_0 \in M$ where all the principal curvatures $k_i(p_0)$ have the same sign, then M^n is a totally umbilical round sphere.*

Proof. Since there exists a point $p_0 \in M$ where all the principal curvatures have the same sign, we may assume, by choosing the appropriate orientation on M^n, that $k_i(p_0) < 0$, for $i = 1, \ldots, n$. Our objective now is to see that $H_1H_r - H_{r+1} \geq 0$ everywhere on M^n, equality holding at the umbilical points. To do that, we will follow the ideas of Montiel and Ros [13, Lemma 1] and their use of Garding inequalities [5]. Actually, from the proof of Lemma 1 in [13] and taking into account the sign convention in our definition of H_i given in Section 2, it follows that the constant $H_r = H_r(p_0)$ is positive with

$$H_{r-1}(p) \geq H_r^{(r-1)/r} > 0 \quad \text{for all } p \in M, \tag{15}$$

and that

$$H_1 \geq H_{r-1}^{1/(r-1)} \tag{16}$$

on M. Moreover, the equality in the above inequalities happens only at umbilical points. From (13) we also know that $H_r^2 - H_{r-1}H_{r+1} \geq 0$, so that

$$H_{r+1} \leq \frac{H_r^2}{H_{r-1}}.$$

This implies, jointly with (15) and (16), that

$$\begin{aligned} H_1H_r - H_{r+1} &\geq \frac{H_r}{H_{r-1}}(H_1H_{r-1} - H_r) \\ &\geq \frac{H_r}{H_{r-1}}(H_1H_{r-1} - H_{r-1}^{r/(r-1)}) \\ &= H_r(H_1 - H_{r-1}^{1/(r-1)}) \geq 0, \end{aligned}$$

with equality at the umbilical points of the hypersurface. Jointly with the integral formula (14) and the reasoning given after that integral formula finishes the proof of our result. □

Another hypothesis which implies that $H_1 H_r - H_{r+1} \geq 0$, with equality at the umbilical points, is the next one: $H_i > 0$ for $i = 1, \ldots, r$ and H_r constant, with $r \leq n - 1$. Indeed, for each $i = 1, \ldots, r$ we have the inequalities

$$H_i^2 - H_{i-1} H_{i+1} \geq 0,$$

with equality at the umbilical points. Since each $H_i > 0$, this is equivalent to

$$\frac{H_1}{H_0} \geq \frac{H_2}{H_1} \geq \cdots \geq \frac{H_r}{H_{r-1}} \geq \frac{H_{r+1}}{H_r},$$

with equality at any stage only at umbilical points. That is, $H_1 H_r - H_{r+1} \geq 0$, with equality at the umbilical points. This allows us to state the following consequence.

Proposition 6. *Let $\psi : M^n \longrightarrow \mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$ be a compact space-like hypersurface in de Sitter space with $H_1 > 0, \ldots, H_r > 0$ and constant H_r, $2 \leq r \leq n - 1$. Then M^n is a totally umbilical round sphere.*

In order to state an interesting consequence of our results, let us introduce the following terminology. Let $a \in \mathbf{L}^{n+2}$ be a unit time-like vector. The intersection of $\mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$ and the space-like hyperplane $\{x \in \mathbf{L}^{n+2} : \langle a , x \rangle = 0\}$ defines a round n-sphere of radius 1, which is a totally geodesic hypersurface in $\mathbf{S}_1^{n+1}$. We will refer to that sphere as the equator of $\mathbf{S}_1^{n+1}$ determined by a. This equator divides the de Sitter space into two connected components, the future which is given by

$$\{x \in \mathbf{S}_1^{n+1} : \langle a , x \rangle < 0\},$$

and the past, given by

$$\{x \in \mathbf{S}_1^{n+1} : \langle a , x \rangle > 0\}.$$

The reason for this terminology is due to the fact that, in the time-orientation of $\mathbf{S}_1^{n+1}$ determined by a, the subset $\{x \in \mathbf{S}_1^{n+1} : \langle a , x \rangle < 0\}$ represents the events which are in the chronological future of the equator determined by a. Using this terminology, we may state the following result.

Theorem 7. *The only compact space-like hypersurfaces in de Sitter space with constant rth mean curvature H_r, $2 \leq r \leq n$, which are contained in the chronological future (or past) of an equator of $\mathbf{S}_1^{n+1}$ are the totally umbilical round spheres.*

Proof. Let us assume, for instance, that the hypersurface $\psi : M^n \longrightarrow \mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$ is contained in the future of the equator determined by a unit time-like vector $a \in \mathbf{L}^{n+2}$ (the case of the past is similar), and let us orient M^n by the Gauss map N which is in the same time-orientation as a, i.e., $\langle a , N\rangle \le -1 < 0$. Since the height function $\langle a , \psi\rangle$ is negative on M^n, by compactness there exists a point $p_0 \in M$ where it attains its maximum

$$\langle a , \psi(p_0)\rangle = \max_{p\in M}\langle a , \psi(p)\rangle < 0.$$

Therefore,

$$\nabla\langle a , \psi\rangle(p_0) = a^{\mathrm{T}}(p_0) = 0 \tag{17}$$

and from (10),

$$\nabla^2\langle a , \psi\rangle(p_0)(v, w) = -\langle a , \psi(p_0)\rangle\langle v , w\rangle - \langle a , N(p_0)\rangle\langle A_{p_0}(v) , w\rangle \le 0$$

for all $v, w \in T_{p_0}M$. On the other hand, since $\langle a , N\rangle^2 = 1 + \langle a , \psi\rangle^2 + |a^{\mathrm{T}}|^2$, (17) implies that

$$-\langle a , N(p_0)\rangle = \sqrt{1 + \langle a , \psi(p_0)\rangle^2}.$$

Therefore, choosing $\{e_1, \ldots, e_n\}$ as a basis of principal directions at the point p_0 we conclude that

$$k_i(p_0) \le \frac{\langle a , \psi(p_0)\rangle}{\sqrt{1 + \langle a , \psi(p_0)\rangle^2}} < 0$$

for each $i = 1, \ldots, n$. Thus, when $r \le n - 1$ we are under the hypothesis of Theorem 5 and the result follows.

When $r = n$, the constant Gauss–Kronecker curvature $H_n = H_n(p_0)$ is positive, so that we have, as in the proof of Theorem 5, that all the rth mean curvatures are positive on M^n and they satisfy

$$H_1 \ge H_2^{1/2} \ge \cdots \ge H_{n-1}^{1/(n-1)} \ge H_n^{1/n} > 0,$$

with equality at any stage only at umbilical points. In particular, from $\langle a , \psi\rangle < 0$,

$$H_{n-1}\langle a , \psi\rangle \le H_n^{(n-1)/n}\langle a , \psi\rangle.$$

Integrating now this inequality, and using the first and the last Minkowski formulas, we obtain

$$\begin{aligned} H_n \int_M \langle a , N\rangle \, \mathrm{d}V &= \int_M H_{n-1}\langle a , \psi\rangle \, \mathrm{d}V \le \int_M H_n^{(n-1)/n}\langle a , \psi\rangle \, \mathrm{d}V \\ &= H_n^{(n-1)/n} \int_M H_1\langle a , N\rangle \, \mathrm{d}V, \end{aligned}$$

i.e.,

$$\int_M (H_1 - H_n^{1/n})\langle a , N\rangle \, \mathrm{d}V \ge 0,$$

with equality if and only if M^n is totally umbilical. But $H_1 \ge H_n^{1/n}$ and then $(H_1 - H_n^{1/n})\langle a , N\rangle \le 0$, getting the equality and the result. □

Appendix A. Another proof of Minkowski formulas

In this section we will present a more geometric proof of our Minkowski formulas stated in Theorem 2. Let $\psi : M^n \longrightarrow \mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$ be a compact space-like hypersurface in de Sitter space and let $a \in \mathbf{L}^{n+2}$ be a fixed arbitrary vector. The gradient of the height function $\langle a, \psi\rangle$ is $\nabla\langle a, \psi\rangle = a^{\mathrm{T}}$, so that from (10) its Laplacian is given by

$$\Delta\langle a, \psi\rangle = -n\langle a, \psi\rangle - \operatorname{tr}(A)\langle a, N\rangle = -n\langle a, \psi\rangle + nH\langle a, N\rangle,$$

where $H = H_1$ is the mean curvature of the immersion ψ. Integrating this on M^n we have that

$$\int_M (-\langle a, \psi\rangle + H\langle a, N\rangle)\,\mathrm{d}V = 0, \tag{A.1}$$

which is nothing but the first Minkowski formula.

Let us consider for $t \in \mathbb{R}$ the parallel hypersurface $\psi_t : M^n \longrightarrow \mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$, which is given by

$$\psi_t(p) = \overline{\exp}_{\psi(p)}(tN(p)) = \cosh(t)\psi(p) + \sinh(t)N(p), \quad p \in M,$$

where $\overline{\exp}$ denotes the exponential map in $\mathbf{S}_1^{n+1}$. Since M^n is compact and $\psi_0 = \psi$ is space-like, there exists $\varepsilon > 0$ such that each ψ_t is a space-like hypersurface for $|t| < \varepsilon$. Since (A.1) holds for any space-like hypersurface, then for $|t| < \varepsilon$ we obtain that

$$\int_M (-\langle a, \psi_t\rangle + H_t\langle a, N_t\rangle)\,\mathrm{d}V_t = 0, \tag{A.2}$$

where H_t is the mean curvature of ψ_t with respect to the orientation given by its Gauss map N_t and $\mathrm{d}V_t$ is the volume element of M^n with respect to the metric induced by ψ_t and the chosen orientation. Our objective now is to compute explicitly the quantities appearing in (A.2). A direct calculation gives

$$(\mathrm{d}\psi_t)_p(v) = \mathrm{d}\psi_p(\cosh(t)v - \sinh(t)A_p(v)), \tag{A.3}$$

for any $p \in M^n$ and $v \in T_pM$, which implies that

$$N_t = \sinh(t)\psi + \cosh(t)N, \tag{A.4}$$

and

$$\mathrm{d}V_t = \cosh^n(t)P(\tanh(t))\,\mathrm{d}V, \tag{A.5}$$

where

$$P(T) = \prod_{i=1}^{n}(1 - k_iT) = \sum_{i=0}^{n}\binom{n}{i}H_iT^i.$$

On the other hand, differentiating (A.4) and using (A.3) we have that

$$\sinh(t)v - \cosh(t)A_p(v) = -\cosh(t)(A_t)_p(v) + \sinh(t)A_p((A_t)_p(v)),$$

for any $p \in M^n$ and $v \in T_pM$, where A_t is the shape operator of ψ_t associated to N_t. This implies that if $\{e_1, \ldots, e_n\}$ is a basis of principal directions at the point p for the immersion ψ satisfying $A_p(e_i) = k_i(p)e_i$, then $\{e_1, \ldots, e_n\}$ is also a basis of principal directions at p for the immersion ψ_t satisfying

$$(A_t)_p(e_i) = \frac{\tanh(t) - k_i(p)}{-1 + k_i(p)\tanh(t)}e_i.$$

In particular,

$$H_t = -\frac{1}{n}\,\mathrm{tr}(A_t) = \frac{P'(\tanh(t)) + nP(\tanh(t))\cosh(t)\sinh(t)}{nP(\tanh(t))\cosh^2(t)}. \tag{A.6}$$

Using (A.4)–(A.6) and the definition of ψ_t, Eq. (A.2) becomes equivalent to

$$\sum_{i=0}^{n-1}(n-i)\binom{n}{i}\tanh^i(t)\int_M(-H_i\langle a\,,\psi\rangle + H_{i+1}\langle a\,,N\rangle)\,\mathrm{d}V = 0,$$

which is a polynomial equation in the variable $\tanh(t)$ vanishing for all $|t| < \varepsilon$. Therefore, all its coefficients must vanish, and these coefficients are, up to a constant, precisely

$$\int_M(-H_i\langle a\,,\psi\rangle + H_{i+1}\langle a\,,N\rangle)\,\mathrm{d}V, \quad i = 0, \ldots, n-1.$$

Acknowledgements

Luis J. Alías and Juan A Aledo were partially supported by DGICYT grant no. PB97-0784-C03-02 and Consejería de Educación y Cultura CARM grant no. PB/5/FS/97 MAT, Programa Séneca (PRIDTYC), while Alfonso Romero was partially supported by DGICYT grant no. PB97-0784-C03-01.

References

[1] K. Akutagawa, On space-like hypersurfaces with constant mean curvature in the de Sitter space, Math. Z. 196 (1987) 13–19.

[2] Q.-M. Cheng, S. Ishikawa, Space-like hypersurfaces with constant scalar curvature, Manuscripta Math. 95 (1998) 499–505.

[3] S.Y. Cheng, S.T. Yau, Hypersurfaces with constant scalar curvature, Math. Ann. 225 (1977) 195–204.

[4] Y. Choquet-Bruhat, J. York, The Cauchy problem, in: A. Held (Ed.), General Relativity and Gravitation, Plenum Press, New York, 1980.

[5] L. Garding, An inequality for hyperbolic polynomials, J. Math. Mech. 8 (1959) 957–965.

[6] A.J. Goddard, Some remarks on the existence of space-like hypersurfaces of constant mean curvature, Math. Proc. Cambridge Phil. Soc. 82 (1977) 489–495.

[7] G. Hardy, J.E. Littlewood, G. Póyla, Inequalities, 2nd ed., Cambridge Mathematical Library, Cambridge, 1989.
[8] C.C. Hsiung, Some integral formulas for closed hypersurfaces, Math. Scand. 2 (1954) 286–294.
[9] H. Li, Global rigidity theorems of hypersurfaces, Ark. Mat. 35 (1997) 327–351.
[10] A. Lichnerowicz, L'integration des equations de la gravitation relativiste et le probleme des n corps, J. Math. Pures Appl. 23 (1944) 37–63.
[11] J.E. Marsden, F.J. Tipler, Maximal hypersurfaces and foliations of constant mean curvature in general relativity, Phys. Rep. 66 (1980) 109–139.
[12] S. Montiel, An integral inequality for compact space-like hypersurfaces in de Sitter space and applications to the case of constant mean curvature, Indiana Univ. Math. J. 37 (1988) 909–917.
[13] S. Montiel, A. Ros, Compact hypersurfaces: the Alexandrov theorem for higher order mean curvatures, in: B. Lawson, K. Tenenblat (Eds.), Differential Geometry, Longman, Essex, 1991, pp. 279–296.
[14] J. Ramanathan, Complete space-like hypersurfaces of constant mean curvature in de Sitter space, Indiana Univ. Math. J. 36 (1987) 349–359.
[15] R.C. Reilly, Variational properties of functions of the mean curvature for hypersurfaces in space forms, J. Differential Geom. 8 (1973) 465–477.
[16] H. Rosenberg, Hypersurfaces of constant curvature in space forms, Bull. Sci. Math. 117 (1993) 211–239.
[17] Y. Zheng, On space-like hypersurfaces in the de Sitter space, Ann. Global Anal. Geom. 13 (1995) 317–321.
[18] Y. Zheng, Space-like hypersurfaces with constant scalar curvature in the de Sitter spaces, Differential Geom. Appl. 6 (1996) 51–54.